CONJUGACY PROBLEM IN GROUPS OF NON-ORIENTED GEOMETRIZABLE 3-MANIFOLDS

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Abstract. We have proved in [Pr] that fundamental groups of oriented geometrizable 3-manifolds have a solvable conjugacy problem. We now focus on groups of non-oriented geometrizable 3-manifolds in order to conclude that all groups of geometrizable 3-manifolds have a solvable conjugacy problem.

Introduction

Since enonced by M.Dehn in the early 1910's the word problem and conjugacy problem in finitely presented groups have become fundamental problems in combinatorial group theory. Following the work of Novikov [No] and further authors on their general unsolvability, it has become fairly natural to ask for any finitely presented group whether it admits a solution or not. For example in [De1, De2, De3], Dehn has solved those problems for fundamental groups of surface ; its motivation was topological.

Given a finite presentation of a group G, a solution to the *word problem* is an algorithm which given $\omega, \omega' \in G$ (as a couple of words on the generators), decides whether $\omega = \omega'$ in G or not. A solution to the *conjugacy problem* is an algorithm which given $u, v \in G$ decides whether $\exists\ h \in G$ such that $u = hvh^{-1}$ in G or not.

It turns out that existence of solutions does not depend of the finite presentation involved. Hence existence of a solution in G to any of these problems only depends on the isomorphism class of G. We say that G has a *solvable* word problem (resp. conjugacy problem) if G admits a solution to the word problem (resp. conjugacy problem).

By a 3-manifold we mean a connected compact manifold with boundary of dimension 3 ; a 3-manifold may be oriented or not. We work in the PL category ; according to the hauptvermutung and Moise theorem this is not restrictive. Following the work of Thurston (cf. [Th]) an oriented 3-manifold M is *geometrizable* if the pieces obtained in its canonical topological decomposition (roughly speaking along essential spheres, discs and tori) have interiors which admit complete locally homogeneous riemanian metrics. A non oriented 3-manifold is said to be geometrizable, if its orientation cover is geometrizable.

In the class of fundamental groups of geometrizable 3-manifolds, the word problem is known to be solvable, following early work of Waldhausen ([Wa]) as well as more recent work of Epstein and Thurston on automatic group theory (cf. [CEHLPT]). We make use of this result in our proof.

We have considered in [Pr] the conjugacy problem in groups of oriented geometrizable 3-manifolds. We are now concerned with the non-oriented case to provide a conclusion concerning the conjugacy problem in the whole class of groups of geometrizable 3-manifolds.

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1. Statement of the result

This work is entirely devoted to the proof of the following result :

Theorem 1.1. *Fundamental groups of non-oriented geometrizable 3-manifolds have solvable conjugacy problem.*

It does not follow as a consequence of existence of a solution in the oriented case (cf. [Pr]), since D.Collins and C.Miller have shown that the conjugacy problem can be unsolvable in a group even when solvable in an index 2 subgroup ([CM]). Nevertheless our strategy will consist essentially in reducing to the conjugacy problem in the oriented case. Together with a solution in the oriented case (cf. [Pr]), one obtains

Theorem 1.2. *Fundamental groups of geometrizable 3-manifolds have solvable conjugacy problem.*

as well as the following corollaries. The former one is a topological rephrasing : a loop γ in M defines a conjugacy class in $\pi_1(M)$ which only depends on its free homotopy class.

Theorem 1.3. *Given a geometrizable 3-manifold M there exists an algorithm which decides for any couple of loops γ, γ' in M whether γ and γ' are freely homotopic.*

Concerning decision problems relative to boundary subgroups, one has :

Theorem 1.4. *Let M a geometrizable 3-manifold M, F any compact connected surface lying in ∂M and $G = \pi_1(M)$, $H = i_*(\pi_1(F))$; there exists an algorithm which decides for any $g \in G$ whether $g \in H$ (resp. g is conjugate to an element of H).*

And one has the topological rephrasing :

Theorem 1.5. *Given any 3-manifold M with $\partial M \neq 0$, and F any connected surface in ∂M, one can decide for any loop γ (resp. $*$-based loop, $* \in F$) in M whether up to homotopy (resp. homotopy with $*$ fixed) γ lies in F.*

2. Some decision results in extensions by $\mathbb{Z}_2$

Let G a group and $u, v, h \in G$. Once $u = hvh^{-1}$ we shall use the notation $u = v^h$ or $u \sim v$. We denote $Z_G(v) = \{u \in G \mid uv = vu\}$ the centralizer of v in G and $C_G(u, v) = \{h \in G \mid u = v^h\}$. The subset $C_G(u, v)$ of G is either empty (when $u \not\sim v$) or $h.Z_G(v)$ for some $h \in G$ such that $u = v^h$.

Lemma 2.1. *Let G be a group and H an index 2 subgroup of G with solvable conjugacy problem. Given any couple of elements $u, v \in H$ one can decide whether u and v are conjugate in G.*

Proof. Given a set of representative $a_0 = 1, a_1$ of H/G, in order to decide whether $u, v \in H$ are conjugate in G it suffices to decide whether u is conjugate in H to any of the $a_i v a_i^{-1}$ for $i = 0, 1$. □

Lemma 2.2. *Let G be a group and H be an index 2 subgroup of G ; suppose that H and G have respectively solvable conjugacy problem and solvable word problem. Let $v \in G \backslash H$ such that $Z_G(v) = Z_G(v^2)$; then on can decide for any $u \in G$ whether u and v are conjugate in G.*

Proof. Let $v \in G$ be as above, and suppose one wants to decide for some given $u \in G$ whether $u \sim v$ in G. With the preceding lemma one can decide whether u^2 and v^2 are conjugate in G. If not then u and v are definitely not conjugate in G. So suppose u^2 and v^2 are conjugate in G, say $u = k.v.k^{-1}$ for some $k \in G$ that one can find using a solution to

the word problem in G, so that $C_G(u^2, v^2) = k.Z_G(v^2)$. Obviously $C_G(u, v) \subset C_G(u^2, v^2)$ and moreover since $Z_G(v^2) = Z_G(v)$, if $C_G(u, v)$ is non empty it must equal $C_G(u^2, v^2)$. Hence to decide whether $u \sim v$ in G it suffices to decide with the word problem in G whether $u = v^k$ or not. □

We also remark the following observation.

Lemma 2.3. *Let G be a f.g. group with solvable word problem, and H be a finite index subgroup. Then the generalised word problem of H in G is solvable.*

Proof. Given a set of representative $a_0, a_1, \ldots, a_n$ of G/H and a finite set of generators of H, as well as $g \in G$, one can enumerate all elements $h \in H$ and check in G whether $g = a_i.h$ for some $i = 1, 2, \ldots, n$. This process must terminate to provide the class of g in G/H. □

This last algorithm is far from being efficient, but anyway note that it will be redondant here ; we only give it as an alternative algorithm for algebra enclined readers while in our proof it will replaced by a topological argument.

3. Checking for centralizers in groups of oriented 3-manifolds

Let M be an oriented closed irreducible 3-manifold, and T the set of essential tori in the JSJ decomposition of M (T may be empty). When T is non empty, T is well defined up to isotopy, and the open manifold $M \setminus T$ has connected components which are open manifolds either homeomorphic to Seifert fibered spaces or to finite volume hyperbolic manifolds. There exists a canonical map $r : M \setminus T \longrightarrow M$ which is an embedding. The *Seifert characteristic submanifold* (cf. [JS]) of M is then defined to be the compact submanifold of M with connected components constructed as follows :
(i) For each Seifert fibered space component S_i of $M \setminus T$ cut the ends of $r(S_i)$ and then consider its adherence in M. Such components are called *non trivial Seifert submanifolds*
(ii) For each torus T_i in T which is not parallel in M to a boundary component of a non trivial Seifert submanifold of M consider a regular neighborhood (homeomorphic to $S^1 \times S^1 \times I$) of T so that they are all pairwise disjoint and do not intersect non trivial Seifert submanifolds. Such coponents are called *trivial Seifert submanifolds*.

If T is empty the Seifert characteristic submanifold is defined to be the empty set if M is not a Seifert fibered space, and M otherwise. The Seifert characteristic submanifold is well defined up to isotopy.

The embedding of components of the Seifert characteristic submanifold induces, up to conjugacy, embeddings of their fundamental groups in $\pi_1(M)$. Once embeddings are given, their images are called *Seifert subgroups* of $\pi_1(M)$; they can be isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ (when induced from a trivial Seifert submanifold) and called $\mathbb{Z} \oplus \mathbb{Z}$ *Seifert subgroups* ; otherwise they are fundamental groups of Seifert spaces and called *non* $\mathbb{Z} \oplus \mathbb{Z}$ *Seifert subgroups*.

Now let M be an oriented 3-manifold whose boundary only consists of spheres, and such that the closed 3-manifold $\hat{M}$ obtained by attaching balls to ∂M is irreducible. Depending on a given embedding of M into $\hat{M}$, the Seifert characteristic submanifold of M is defined to be the intersection of the Seifert characteristic submanifold of $\hat{M}$ with M. Its components are punctured Seifert submanifolds and are defined up to the embedding of M. We make use of the terminology of trivial (resp. non trivial) Seifert submanifolds of M for components which come from trivial (resp. non trivial) Seifert submanifolds of $\hat{M}$. We also define, up to conjugacy, Seifert subgroups of $\pi_1(M) = \pi_1(\hat{M})$.

Lemma 3.1. *Let M be an oriented 3-manifold such that $\hat{M}$ is closed irreducible. Suppose a triangulation of M as well as an embedding of M into $\hat{M}$ are given. Then one can algorithmically construct the Seifert characteristic submanifold of M and recognize trivial Seifert submanifolds from other Seifert submanifolds. Given in $\pi_1(M)$ an element u and a Seifert subgroup K, one can decide whether u is conjugate to an element of K and, if so, produce a conjugating element.*

Proof. Glue PL-balls to ∂M in order to obtain $\hat{M}$ together with a triangulation. Then apply the algorithm of [JT] in order to obtain the JSJ decomposition T of $\hat{M}$ as well as the Seifert characteristic submanifold. Note that the algorithm also produces Seifert invariants of the Seifert submanifolds so that one can recognize trivial one from the others. Since the embedding of M into $\hat{M}$ is given it provides the Seifert characteristic submanifold of M. Note that $\hat{M}$ is a $S^1 \times S^1$-bundle over S^1 modelled on geometry Sol if and only if $M \setminus T$ is homeomorphic to the interior of $S^1 \times S^1 \times I$, which can be easily checked.

The JSJ decomposition of $\hat{M}$ splits $\pi_1(M) = \pi_1(\hat{M})$ into a fundamental group of a graph of group. One can produce such a graph of group as well as a finite presentation of $\pi_1(M)$ and embeddings of edge and vertex groups in $\pi_1(M)$, so that $\mathbb{Z} \oplus \mathbb{Z}$-Seifert subgroups (resp. non $\mathbb{Z} \oplus \mathbb{Z}$ Seifert subgroups) are edge subgroups (resp. vertex subgroups) of $\pi_1(M)$. In the following we exlusively make use of results established in [Pr]. Given an element $u \in \pi_1(M)$ one can use corollary 4.1 of [Pr] to change u into a cyclically reduced conjugate according to this splitting (§3.1, [Pr]) ; it also provides the conjugating element. If $\hat{M}$ is a $S^1 \times S^1$-bundle over S^1 modelled on geometry Sol, then obviously u is conjugate to $K = \mathbb{Z} \oplus \mathbb{Z}$ if and only if u lies in the vertex subgroup K. Otherwise, it follows from theorem 3.1, lemma 2.2 and proposition 4.1 of [Pr] that u lies in a given vertex subgroup G_V, for some vertex V, and that u is conjugate in $\pi_1(M)$ to an element of K exactly if one of the following cases occur : (i) $G_V = K$; (ii) K is an edge subgroup of G_V, and u is conjugate in G_V to an element of K ; (iii) K is a vertex subgroup $G_{V'}$, for some vertex V' and u is conjugate in G_V to an element lying in an edge subgroup G_e for some edge e going from V to V'. In case (i) or Sol geometry a solution is obvious, while cases (ii) and (iii) reduce to deciding whether u is conjugate in G_V to a given edge subgroup. One applies the algorithms of [Pr] given in proposition 5.1 (if G_V comes from a Seifert component) or theorem 6.3 (otherwise) to decide so. Note that going into the line of the proofs on sees that the algorithms produce, if any, a conjugating element ; nevertheless one can also obviously find such an element by a naive algorithm. □

Remark : Jaco and Shalen have proved (theorem VI.I.6, [JS]) that whenever G is the group of an oriented closed Haken manifold an element $g \in G$ as either a cyclic centralizer or up to conjugacy its centralizer lies in a Seifert subgroup. If M is an oriented irreducible non Haken manifold, M must be closed and homeomorphic to either a Seifert space or an hyperbolic manifold, and it follows that the centralizers have the same structure as in the Haken case. If ∂M contains spheres and $\hat{M}$ is irreducible the same conclusion applies. Hence in such cases only the following can occur :
(i) $Z_G(g) = \mathbb{Z}$,
(ii) g is conjugate to a $\mathbb{Z} \oplus \mathbb{Z}$ Seifert subgroup, $Z_G(g) = \mathbb{Z} \oplus \mathbb{Z}$,
(iii) g is conjugate to a non $\mathbb{Z} \oplus \mathbb{Z}$ Seifert subgroup.
So that the last lemma allows one to decide for any element g in the group of such a triangulated 3-manifold whether cases (i), (ii) or (iii) occur and provide, if any, the Seifert characteristic submanifold S as well as a Seifert subgroup $K = i_*(\pi_1(S)) \subset \pi_1(M)$ containing an element conjugate with g, together with a conjugating element.

4. Proof of the main result

We are now ready to give proofs of the results enonced in §1. We are first concerned with the main result, that is theorem 1.1. The following preliminary step reduces the proof to the case of closed irreducible geometrizable 3-manifolds.

Lemma 4.1. *Conjugacy problem in groups of non-oriented geometrizable 3-manifolds reduces to conjugacy problem in groups of closed irreducible geometrizable 3-manifolds. Moreover if a triangulation of M as well as a solution to the word problem in $\pi_1(M)$ are given, the reduction process applied to $\pi_1(M)$ is constructive.*

Proof. Let M be a non-oriented geometrizable 3-manifold ; we are concerned with the conjugacy problem in $\pi_1(M)$. We process the reduction in two steps.

Gluing a 3-ball to each spherical component of ∂M leaves $\pi_1(M)$ inchanged ; so we suppose in the following that M has no spherical boundary component. If ∂M is non-empty, double the manifold M along its boundary to obtain the closed non-oriented 3-manifold that we shall denote $2M$. Lemma 1.1 of [Pr] asserts that the inclusion map of M in $2M$ induces an embedding of $\pi_1(M)$ in $\pi_1(2M)$, and that $u, v \in \pi_1(M)$ are conjugate in $\pi_1(M)$ if and only if they are conjugate in $\pi_1(2M)$; hence the conjugacy problem in $\pi_1(M)$ reduces to the conjugacy problem in $\pi_1(2M)$. Moreover the closed manifold $2M$ is geometrizable ; indeed if $\overline{M}$ and $\overline{2M}$ denote respectively the orientation covers of M and $2M$, one has that $\overline{2M}$ is the double of $\overline{M}$, and since M is geometrizable, lemma 1.2 of [Pr] states that $\overline{2M}$ is geometrizable. Hence the conjugacy problem in $\pi_1(M)$ reduces to conjugacy problem in groups of closed geometrizable 3-manifolds. Moreover if M is given by a triangulation, the reduction can be achieved in a constructive way for one can constructively produce a triangulation of $2M$ and the natural embedding from $\pi_1(M)$ to $\pi_1(2M)$.

We are now concerned with the second step, and suppose M to be moreover closed. A Kneser-Milnor decomposition splits M in a connected sum of the prime closed geometrizable (non necessarily non-oriented) factors $M_1, M_2, \dots, M_i$ and $\pi_1(M)$ splits as the free product of the $\pi_1(M_1), \pi_1(M_2), \dots, \pi_1(M_n)$. Basic fact upon conjugacy in free products ([MKS]) shows that the conjugacy problem in $\pi_1(M)$ reduces to conjugacy problems in each of the $\pi_1(M_i)$. Now either M_i is a S^2-bundle over S^1, and hence $\pi_1(M_i) = \mathbb{Z}$, or M_i is irreducible. Hence conjugacy problem in $\pi_1(M)$ reduces to conjugacy problem in groups of closed irreducible geometrizable 3-manifolds. If M is given by a triangulation an algorithm in [JT] for a Kneser-Milnor decomposition allows to process the reduction in a constructive way. □

We are now ready to show the main result upon conjugacy problem in non oriented geometrizable manifolds. Note that we not only prove the existence of a solution : we rather show how, given a triangulation of M, one can construct the algorithm.

Proof of theorem 1.1. According to lemma 4.1, and to a solution to the conjugacy problem in groups of oriented geometrizable 3-manifolds ([Pr]), we are left with the case of groups of closed irreducible non-oriented geometrizable 3-manifolds. In the following M stands for a closed irreducible non-oriented geometrizable 3-manifold, and $p : N \longrightarrow M$ for the orientation cover of M.

Lemma 4.2. *Given a triangulation of M one can algorithmically produce a triangulation of its orientation cover N as well as the covering map and covering automorphism.*

Proof of lemma 4.2. The triangulation of M can be easily given as a triangulation of a PL-ball B together with a gluing of pairs of triangles in ∂B. Pick an orientation of B ; it induces an orientation of each triangle in ∂B. Identify paired triangles in ∂B each time

their gluing preserves orientation, to obtain a new oriented PL-manifold C together with orientation reversing gluing of pairs of triangles in ∂C. Consider a copy C' of C and give C' the reverse orientation. For each triangle t in ∂C denote by t' its copy in $\partial C'$. For each gluing of triangles t_1, t_2 in ∂C, glue coherently in $C \cup C'$, t_1 with t_2' and t_1' with t_2. The manifold obtained is N together with a triangulation, and the construction implicitly produces the covering map $p : N \longrightarrow M$ as well a the covering automorphism. □

We suppose in the following M to be given by a triangulation. The above lemma allows to produce a triangulation of N as well as the covering map $p : N \longrightarrow N$ and the covering automorphism σ. Let $G = \pi_1(M)$ and $H = \pi_1(N)$.

The oriented manifold N may be reducible. In such a case (cf. [Sw]), N contains a minimal system of essential pairwise disjoint σ-invariant spheres $S = \{S_1, S_2, \dots, S_q\}$, with image in M a system of essential pairwise disjoint projective planes $P = \{P_1, P_2, \dots, P_q\}$ such that : (i) N cutted along S decomposes into components $N_1, N_2, \dots, N_p$ and n components homeomorphic to $S^2 \times S^1$, such that each manifold $\hat{N}_i$ obtained by gluing a ball to each $S^2 \subset \partial N_i$ is irreducible and non simply connected ; (ii) $\pi_1(N) = \pi_1(N_1) * \cdots * \pi_1(N_p) * F(n)$; (iii) M cutted along P has component $M_1, M_2, \dots, M_p$ as well as components $\mathbb{P}_2 \times I$, and the covering projection sends N_i onto M_i and each $S^2 \times I$ component onto a $\mathbb{P}_2 \times I$ component ; (iv) $\pi_1(M)$ splits as a graph of group where vertex groups are $\pi_1(M_1), \dots, \pi_1(M_p)$ as well as n groups of order 2 and edge groups all have order 2. Apply the following algorithm to find a system S of pairwise disjoint essential σ-invariant spheres.

Lemma 4.3. *One can algorithmically find a system of pairwise disjoint essential σ-invariant spheres in N and a system of pairwise disjoint essential projective planes in M, as above.*

Proof of lemma 4.3. Apply in N the algorithm of [JT] to find a system S of disjoint spheres which decomposes N into pieces which, once balls are glued on the boundary, are all irreducible. Apply the S^3 recognition algorithm to detect all pieces in $N \setminus S$ homeomorphic to a ball or to $S^2 \times I$. Delete each S_1 in S which bounds either a ball or two $S^2 \times I$ pieces, each each couple S_1, S_2 in S whenever they are both separating and bound a $S^2 \times I$ piece. Up to this point S consists only of essential spheres. It remains to deform S so that the spheres be σ-invariant. For each $S_1 \in S$, $\sigma(S_1) \cap S_1 \neq \emptyset$ cause otherwise N wouldn't be irreducible. For each $S_1 \in S$ construct $\sigma(S_1)$, and use the S^3 recognition algorithm to construct an essential sphere included in $S_1 \cup \sigma(S_1)$ which is preserved under σ ; change in S, S_1 into this sphere. Each time two spheres in S are not disjoint apply an analog process to change them into disjoint essential spheres ; since the number of self intersection of $S \cup \sigma(S)$ decreases the process must stop ; we are finally led with a system of pairwise disjoint essential σ-invariant spheres in N ; their image under the covering projection gives the system P of essential pairwise disjoint projective planes in M. □

Compute finite presentations of G and H in such a way that if S is non empty the generators of H (respectively G) is the union of the generators of the factors in its free product decomposition (resp. generators of the vertex groups plus additional stable letters). Compute the embedding of H in G (by the image of the generators of H in G). Consider also solutions to the word problem in G and H, as well as a solution to the conjugacy problem in H. They can be given in a contructive way : an algorithm for the word problem in G comes from the automatic group theory and can be constructed (cf. [CEHLPT]) ; [Pr] allows to construct an algorithm for the conjugacy problem (and hence the word problem) in H.

Use the solution to the word problem to decide whether G is abelian ; if so a solution to the word problem gives a solution to the conjugacy problem in G. So in the following we suppose that M is not homeomorphic to $\mathbb{P}_2 \times S^1$.

Suppose u and $v \in G$ are given and one wants to decide whether $u \sim v$ in G. Find first respective classes of u, v in G/H. This can be done by using the naive algorithm of lemma 2.3 or can be achieved in a more efficient way by the following process : suppose one knows for each generator a of G whether a is orientation reversing or not. Then u lies in H if and only if the word representing u contains an even number of orientation reversing generators. Note that from a triangulation of M one can easily deduce a generating set of G and check for each generator whether it reverses orientation or not ; for example by constructing a triangulation of M as appearing in the proof of lemma 4.2 : the gluing maps give generators of G.

Since H has index 2 in G, if u and v lie in distinct classes they are definitely not conjugate in G. If u and v both lie in H, then the lemma 2.1 together with a solution to the conjugacy problem in H allow constructively to decide whether u and v are conjugate in G. So in the following we will suppose that u and v both lie in $G \setminus H$.

Use a solution to the word problem in H to decide whether $u^2 = 1$ and $v^2 = 1$. If exactly one of the relations occurs then u and v are not conjugate in G, while if both relations occur then the following lemma allows to constructively decide whether u and v are conjugate or not.

Lemma 4.4. *Let G be as above. One can decide for any couple of order 2 elements $u, v \in G$ whether u and v are conjugate in G.*

Proof. Consider the above system P of essential projective planes in M and delete in P one of P_i, P_j every time P_i, P_j cobound a $\mathbb{P}_2 \times I$ component. It follows from [Ep], [St], [Sw] that each order 2 element in G is conjugate to some $H_i = i_*(\pi_1(P_i)) = \mathbb{Z}_2$ for $P_i \in P$, and moreover if we denote h_i the generator of H_i, the different h_i have disjoint conjugacy classes in G. Hence a naive algorithm goes as follows : the different h_i are given. Enumerate all the elements g of G (as words on a given finite generating set), and in parallel for each g obtained and each h_i use a solution to the word problem in G to decide whether $u = h_i^g$ or $v = h_j^g$. The algorithm halts once it finds such h_i, h_j in the respective conjugacy classes of u and v. Then $u \sim v$ if and only if $h_i = h_j$. □

We will be concerned in the following with the remaining case : u, v both lie in $G \setminus H$ and both have order different than 2 ; according to [Ep] both u and v must have infinite order in G.

Use the lemma 2.1 together with a solution to the conjugacy problem in H to decide whether $u^2 \sim v^2$ in G. If it does not arise then u and v are not conjugate in G. So we suppose in the following that $u^2 \sim v^2$ in G. Find an element $h \in G$ such that $u^2 = (v^2)^h$ in G. This can be naively performed by enumerating all $g \in G$ and for each g obtained by deciding in parallel with a solution to the word problem whether $u^2 = (v^2)^g$; it can also be more efficiently achieved by going into the line of the proofs of [Pr] to remark that the solution to the conjugacy problem provides such a conjugating element h.

Up to this point the set $C_G(u^2, v^2) = h.Z_G(v^2)$ is non empty. It obviously contains the set $C_G(u, v)$; note also that $Z_G(v^2)$ contains $Z_G(v)$ as a subgroup as well as $Z_H(v^2)$ as an index 2 subgroup ; $Z_G(v^2)$ is generated by $Z_H(v^2)$ and v.

We are now interested in the centralizer $Z_H(v^2)$. If the system S of essential spheres in N is non-empty, H splits non trivially as a free product. If so make use of a solution to the word problem in H (together with elementary facts about free products, cf. [MKS]) to write down respective cyclic conjugates dv^2d^{-1} of v^2 and cu^2c^{-1} of u^2 in a cyclically reduced form. If dv^2d^{-1} has length greater than 1 (according to the splitting) then $Z_H(v^2)$

is infinite cyclic. If $Z_H(v^2)$ is non cyclic dv^2d^{-1} and cu^2c^{-1} must both lie in a factor $\pi_1(N_1)$ of the free product decomposition of H. We can apply in N_1 lemma 3.1 together with its following remark to decide whether dv^2d^{-1}, and hence v^2, has a cyclic or $\mathbb{Z}\oplus\mathbb{Z}$ centralizer, or other whether dv^2d^{-1} is conjugate to some non $\mathbb{Z}\oplus\mathbb{Z}$ Seifert subgroup. Note that if dv^2d^{-1} has a non cyclic centralizer, it is conjugate to a Seifert subgroup W of $\pi_1(N_1)$, and the algorithm produces both W and $e \in \pi_1(N_1)$ such that $(ed)v^2(ed)^{-1} \in W$.

We consider first the case where $Z_H(v^2)$ is infinite cyclic ; then $Z_G(v^2)$ contains $\mathbb{Z}$ as an index 2 subgroup. If $Z_G(v^2)$ is torsion-free then it must be cyclic, say $Z_G(v^2) =< w >$. But since $v \in Z_G(v^2)$, v is a power of w, and since $Z_G(v) \subset Z_G(v^2)$, it implies that $Z_G(v) = Z_G(v^2) =< w >$. If $Z_G(v^2)$ has torsion, let us denote by t a generator of its index 2 subgroup $Z_H(v^2)$. The group $Z_G(v^2)$ is generated by v and t and must be one of the groups appearing in the following lemma.

Lemma 4.5. *A torsion group K with generators v, t, such that $< t >\approx \mathbb{Z}$ has index 2 in K must be one of :*

$$< v, t \mid [v, t] = 1, v^2 = t^{2n} >\approx \mathbb{Z}\oplus\mathbb{Z}_2$$
$$< v, t \mid t^v = t^{-1}, v^2 = 1 >\approx \mathbb{Z}_2 * \mathbb{Z}_2$$

Proof of lemma 4.5. The group K admits the presentation $< v, t \mid t^v = t^{\pm 1}, v^2 = t^p >$ with $p \in \mathbb{Z}$; $K\setminus < t >$ contains an element w with finite order $k \neq 0$. In particulary w^k lies in $< t >$ so that k must be even ; hence K contains an element $t^a v$ with order 2. Suppose first that $t^v = t$, so that $1 = (t^a v)^2 = t^{2a+p}$. It follows that p is even, say $p = 2n$, which gives the first above presentation. Suppose then that $t^v = t^{-1}$; one has $1 = (t^a v)^2 = v^2$ which gives the second above presentation. □

The latter group cannot occur since v has infinite order. For the former group, since $[t, v] = 1$, one has $Z_G(v) = Z_G(v^2)$. Hence whenever $Z_H(v^2)$ is infinite cyclic, $Z_G(v) = Z_G(v^2)$ and the lemma 2.2 allows to decide whether $u \sim v$ in G.

We consider now the case where v^2 is conjugate to a Seifert subgroup. We change u, v, h respectively into cuc^{-1}, $(ed)v(ed)^{-1}$ and $ch(ed)^{-1}$ (c, d, e have been constructed above), so that u^2, v^2, h lie in the factor $\pi_1(N_1)$ of the free product decomposition of H, and $u^2 = (v^2)^h$. Moreover v^2 lies in a given Seifert subgroup W of $\pi_1(N_1)$. According to the following lemma we then have $u, v \in \pi_1(M_1)$, where $p(N_1) = M_1$.

Lemma 4.6. *Suppose that K splits as a fundamental group of a graph of group, and that xv^2x^{-1} lies in a vertex subgroup G_V of K. Then xvx^{-1} also lies in G_V.*

Proof of lemma 4.6. Delete an edge in the graph of group of K. It decomposes K in an amalgam or an HNN extension according to whether the edge is separating or not. Denote by K' a factor in this decomposition of K. If $w \in K$ is cyclically reduced and if w^2 lies in the factor K' then w must also lie in K' (cf. [MKS], [Ro]). If w is non cyclically reduced, then it can be written in a reduced form as $w = w_1 \cdots w_k w' w_k^{-1} \cdots w_1^{-1}$ where w' is cyclically reduced. Hence $w^2 = w_1 \cdots w_k (w')^2 w_k^{-1} \cdots w_1^{-1}$ is reduced and non cyclically reduced and cannot lie in the factor K' of K. Note that K' splits as a fundamental group of a graph of group such that its vertex subgroups are vertex subgroups of K. Pursue inductively the process by decomposing K' along an edge until we are led with factors which are vertex subgroups of K. To conclude remark that if xv^2x^{-1} lies in a vertex subgroups G_V of K then, for any of the above decompositions, its reduced form is also cyclically reduced and xv^2x^{-1} must lie in some factor ; the above argument shows that the same conclusion applies to xvx^{-1}, so that xvx^{-1} must lie in G_V. □

Suppose first that v^2 lies in a $\mathbb{Z}\oplus\mathbb{Z}$ Seifert subgroup of $\pi_1(N_1)$, so that $Z_H(v^2) = \mathbb{Z}\oplus\mathbb{Z}$. The element v obviously normalizes $Z_H(v^2)$; hence a $\mathbb{Z}\oplus\mathbb{Z}$ Seifert submanifold

homeomorphic to (an eventually punctured) $T_1 = S^1 \times S^1 \times I$ of N_1 can be chosen to be preserved under the orientation reversing covering automorphism associated with v and its image in M_1 under the covering projection induces the embedding of $Z_G(v^2)$ (generated by $Z_H(v^2)$ and v) ; $Z_G(v^2)$ can only be the group of the Klein Bottle. One must have :

$$Z_H(v^2) =< a, b \mid [a, b] = 1 > \quad Z_G(v^2) =< a, b, t \mid [a, b] = 1, t^2 = a, b^t = b^{-1} >$$

Moreover it's fairly easy to find generators a, b, t of $Z_G(v^2)$ as above : pick an arbitrary base of $Z_H(v^2) = i_*(\pi_1(T_1)) \approx \mathbb{Z} \oplus \mathbb{Z}$, and using a solution to the word problem write down the conjugacy action of v on $Z_H(v^2)$ as a matrix $\mathcal{M}$ in $GL(2, \mathbb{Z})$. Diagonalize $\mathcal{M}$, it has eigen values $l_1 = 1$ and $l_2 = -1$; a, b are the eigenvectors respectively associated with l_1 and l_2. Then use a solution to the word problem in order to write down the generator $t = a^n b^m v$.

Lemma 4.7. *Let $K =< a, b, t \mid [a, b] = 1, t^2 = a, b^t = b^{-1} >$ and $v = a^{n_1} b^{n_2} t$. Then $Z_K(v) =< b^{n_2} t > \supset < a >$ and $v' \sim v$ if and only if $v' = a^{n_1} b^{m_2} t$ with $m_2 = n_2 \mod 2$.*

Proof of lemma 4.7. Let A be the abelian subgroup generated by a and b. Let w be an element of A ; it's easy to see that if w lies in $< a >$ then $Z_K(w) = K$ (a and t commute) and otherwise $Z_K(w) = A$. Let $v = a^{n_1} b^{n_2} t$; with the above $Z_K(v) \cap A =< a >$. Let $z = a^{m_1} b^{m_2} t \in K \setminus A$. Computation shows that $vz = a^{n_1+m_1+1} b^{n_2-m_2}$ and $zv = a^{n_1+m_1+1} b^{m_2-n_2}$, and hence $z \in Z_K(v)$ if and only if $m_2 = n_2$ that is $z = va^{m_1-n_1}$ for some $m_1 \in \mathbb{Z}$. It follows that $Z_K(v)$ is generated by $b^{n_2} t$ and a ; but since $(b^{n_2} t)^2 = a$, $Z_K(v) =< b^{n_2} t >$. We are now concerned with the conjugacy class of v. One has that a and v commute, and for $n \in \mathbb{Z}$, $b^n v b^{-n} = a^{n_1} b^{n_2+2n} t$, while $t^n v t^{-n}$ equals v when n is even and $a^{n_1} b^{-n_2} t$ otherwise. Hence v' and v are conjugate if and only if $v' = a^{n_1} b^{m_2} t$ with $m_2 = n_2 \mod 2$. □

Write down v on the generators a, b, t, say $v = a^{n_1} b^{n_2} t$; with the above lemma $Z_G(v) =< b^{n_2} t > \supset < a >$, and v' is conjugate in $Z_G(v^2)$ to v if and only if $v' = a^{n_1} b^{m_2} t$ with $m_2 = n_2 \mod 2$.

The set $C_G(u^2, v^2) = hZ_G(v^2)$ is given and contains $C_G(u, v)$. Hence u and v are conjugate in G if and only if there exists an element $w \in Z_G(v^2) \setminus Z_G(v)$ such that $h^{-1} u h = w v w^{-1}$. With the above, $u \sim v$ if and only if $h^{-1} u h = a^{n_1} b^{m_2} t$ with $m_2 = n_2 \mod 2$, that is if and only if $h^{-1} u h$ lies in $v < b^2 >$. Use a solution to the word problem in G to decide first whether $h^{-1} u h$ lies in $Z_G(v^2)$, and if yes to secondly write $h^{-1} u h v^{-1}$ on the generators a, b, t and then check whether $h^{-1} u h v^{-1}$ lies in $< b^2 >$. This process allows to decide whether u and v are conjugate in G other not.

Finally suppose that v^2 lies in a non $\mathbb{Z} \oplus \mathbb{Z}$ Seifert subgroup W of H. One has splittings of $\pi_1(N_1)$ and $\pi_1(M_1)$ as fundamental groups of graphs of group induced by the JSJ-decompositions of the irreducible manifold $\hat{N}_1$ and M_1, so that vertex groups of $\pi_1(N_1)$ are index 1 or 2 subgroups of vertex groups of $\pi_1(M_1)$. The element v^2 lies in a vertex subgroup $H_s = W$ coming from a Seifert piece N' in the JSJ splitting of $\hat{N}_1$. Change u into huh^{-1} so that $u^2 = v^2$. According to the lemma 4.6, u, v lie in a vertex subgroup $G_s = p_*(H_s)$ of $\pi_1(M_1)$, such that $G_s \cap H = H_s$ is an index 2 subgroup of G_s (since $v \notin H$). In fact $G_s = \pi_1(M')$ where M' is a piece in the JSJ decomposition of M_1 and one has the orientation cover $p' : N' \setminus \bigcup B^2 \longrightarrow M'$; the covering automorphism can be extended to an orientation reversing involution of N' with a finite number of fixed points.

On the one hand $C_H(u^2, v^2) = Z_H(v^2)$ is included in H_s (cf. theorem VI.I.6, [JS] or remark §3), and on the other $C_G(u^2, v^2) = Z_G(v^2)$ is generated by $Z_H(v^2)$ and v, and hence is included in G_s. Since $C_G(u^2, v^2) \supset C_G(u, v)$, if u and v are conjugate in G, they must be conjugate in G_s.

The manifold N' is a Seifert fibered space which cannot be modelled on Nil geometry (such manifolds do not admit an orientation reversing involution, cf. [Sc]) and according to [NR] H_s is a biautomatic group. Hence (cf. [CEHLPT]) G_s is also biautomatic and one can construct an algorithm to solve the conjugacy problem in G_s. So that we can decide whether u and v are conjugate in G or not. □

Proof of theorem 1.4. Double the 3-manifold M along the identity on F ; one obtains a 3-manifold $2M$. The argument in the proof of lemma 1.2 of [Pr] as well as the observation that the orientation cover of $2M$ is the double of the orientation cover of M along lifting(s) of F show that $2M$ is geometric. Its group splits into an amalgam of two copies of M along H, say $\Gamma = G *_H G$; given g in the G left factor we note $\bar{g}$ the corresponding element of the G right factor. Since the gluing is along the identity $h = \bar{h}$, one has that $u, \bar{u} \in G$ are equal (resp. conjugate) in G if and only if $u \in H$ (resp. u conjugate in G to some $h \in H$). Hence a solution to the word problem (resp. conjugacy problem) in $\pi_1(2M)$ provides the algorithm. □

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